Maxima Minima for a function of Two variable

Here will discuss the maxima & minima of a function $f(x,y)$ with two variables x, y

NOTATIONS

Let $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$ $s = \frac{\partial^2 f}{\partial x \partial y}$

$t = \frac{\partial^2 f}{\partial y^2}$

NOTE:-1

A function $f(x,y)$ has a relative maximum at (x,y) provided

(i) $f(x,y) \in C^2$

(ii) $f_x = f_y = 0$ or $p = q = 0$ at (x,y)

(iii) $f_{xx} f_{yy} - f_{xy}^2 > 0$ or $rt - s^2 > 0$ at (x,y)

(iv) $f_{xx} < 0$ or $r < 0$ at (x,y)

NOTE:-2

A functim $f(x,y)$ has a relative minimum at (x,y) provided

(i) $f(x,y) \in C^2$

(ii) $f_x = f_y = 0$ or $p = q = 0$ at (x,y)

(iii) $f_{xx} f_{yy} - f_{xy}^2 > 0$ or $rt - s^2 > 0$ at (x,y)

(iv) $f_{xx} > 0$ or $r > 0$ at (x,y)

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NOTE:-3

A function $f(x,y)$ has a saddle point at (x,y) provided

(i) $f(x,y) \in C^2$

(ii) $f_x = f_y = 0$ or $p = q = 0$ at (x,y)
or $rt - s^2 < 0$

(iii) $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (x,y)

Example

Test the function $x^2 - xy + y^4$ for relative maximum, relative minimum and saddle point.

Ans: Here, $f(x,y) = x^2 - xy + y^4$

$f_x = 2x - y$ $\quad f_y = -x + 4y^3$ $\quad f_{xy} = -1$

$f_{xx} = 2$ $\quad f_{yy} = 12y^2$

For extremum we must have

$f_x = 0$ & $f_y = 0$

Thus, $f_x = 0 \Rightarrow 2x - xy = 0 \Rightarrow x = \frac{y}{2}$

$f_y = 0 \Rightarrow -x + 4y^3 = 0 \Rightarrow x = 4y^3$

Thus $\frac{1}{2} y = 4y^3 \Rightarrow 8y^3 - y = 0$

$\Rightarrow y(8y^2 - 1) = 0$

$\Rightarrow y = 0$ or $y = \pm \frac{1}{2\sqrt{2}}$

Corresponding $x = 0$ or $x = \pm \frac{1}{4\sqrt{2}}$

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Hence the possible points of extrema are

$(0,0)$, $\left(\frac{1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ and $\left(-\frac{1}{4\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$

Points	$r = f_{xx}$	$s = f_{xy}$	$t = f_{yy}$	$rt - s^2$
$(0,0)$	2	-1	0	-1
$\left(\frac{1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	2	-1	$\frac{3}{2}$	2
$\left(-\frac{1}{4\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$	2	-1	$\frac{3}{2}$	2

Now, $rt - s^2 < 0$ at the point $(0,0)$. Thus $(0,0)$ is a saddle point.

Again $rt - s^2 > 0$, and $r > 0$ at both the points $\left(\frac{1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ and $\left(-\frac{1}{4\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$

So, the function has a relative minimum at these two points.

The minimum value is

$$\left(\frac{1}{4\sqrt{2}}\right)^2 - \left(\frac{1}{4\sqrt{2}}\right)\left(\frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2\sqrt{2}}\right)^4$$

$$= \frac{1}{32} - \frac{1}{16} + \frac{1}{64} = \frac{2-4+1}{64} = -\frac{1}{64}.$$

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